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RESULTANT REACTIONS IN THE TRANSIENT CONTACT PROBLEM FOR AN ANISOTROPIC ELASTIC HALF-SPACE†

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The resultant contact forces and moments are determined for the initial stage of the interaction between an absolutely rigid smooth convex punch and an anisotropic elastic half-space within the framework of the three-dimensional transient problem, unlike the case of vertical indentation considered in [1].

Various aspects of the initial stage of the interaction between a smooth convex punch and an elastic half-space for an isotropic medium were considered in [2–6].

1. FORMULATION OF THE PROBLEM

Consider a homogeneous anisotropic elastic half-space bounded by the plane $x_3 = 0$. In an orthogonal Cartesian system of coordinates $x_1x_2x_3$ (the x_3 axis is directed into the half-space) its motion is governed by the following equations within the framework of the linear theory of elasticity ($i, j, k, m = 1, 2, 3$)

$$c_{kmij}\frac{\partial^2 u_j}{\partial x_k \partial x_i} = \gamma^2 \ddot{u}_m, \quad \sigma_{km} = c_{kmij}\frac{\partial u_j}{\partial x_i}, \quad \gamma^2 = \frac{\rho}{\rho_*} \tag{1.1}$$

Here $\mathbf{u} = u_j\mathbf{e}_j$ is the displacement vector, $\mathbf{e}_j$ are vectors forming a basis, σ_{km} and c_{kmij} are the components of the stress tensor and the tensor of elasticity constants, p is the density of the medium and differentiation with respect to time τ is denoted by a dot. In (1.1) and henceforth summation is carried out over repeated Latin subscripts from 1 to 3. In these formulae and throughout we use dimensionless variables with the following reference units: length L_*, time L_*/c_*, and mass $\rho_* L_*^3$, where L_*, c_* and ρ_* are, respectively, some characteristic length, velocity and density.

Perturbations of one of the two types below are given on the boundary $x_3 = 0$

$$u_j\big|_{x_3=0} = w_j(x_1, x_2) \in \Omega, \quad \sigma_{j3}\big|_{x_3=0} = 0 (x_1, x_2) \notin \Omega \tag{1.2}$$

or

$$u_3\big|_{x_3=0} = w_3(x_1, x_2) \in \Omega, \quad \sigma_{33}\big|_{x_3=0} = 0 (x_1, x_2) \notin \Omega \tag{1.3}$$
$$\sigma_{13}\big|_{x_3=0} = \sigma_{23}\big|_{x_3=0} = 0$$

The solutions are assumed to be bounded at infinity and the initial conditions correspond to the state of rest

$$\mathbf{u}\big|_{\tau=0} = \dot{\mathbf{u}}\big|_{\tau=0} = 0 \tag{1.4}$$

In the linearized formulation of the contact problem Ω is the flat contact domain obtained by shifting the contact surface between the elastic medium and the punch onto the unperturbed boundary of the half-space. Perturbations (1.2) correspond to rigid coupling conditions and (1.3) to free slipping conditions. For an absolutely rigid punch, w_j can be defined by

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$$w_j = (\mathbf{u}_C + \mathbf{r}_1,\ \mathbf{e}_j),\quad \mathbf{u}_C = u_{Cj}\mathbf{e}_j \tag{1.5}$$

where $\mathbf{u}_C$ is the displacement vector of the centre of mass of the punch and $\mathbf{r}_1$ is the position vector of the bounding surface Π of the punch in the system of coordinates linked to it.

The motion of the punch can be described by the well-known initial-value problem for an absolutely rigid body given, for example, in [5]. The velocity of the points of the boundary surface Π can be found as follows:

$$\mathbf{v}_1 \dot{\mathbf{u}}_C = [\boldsymbol{\omega},\ \mathbf{r}_1],\quad \boldsymbol{\omega} = \omega_1\mathbf{e}_i \tag{1.6}$$

where ω is the angular velocity vector of the punch.

The coupling in the contact problem is determined by the presence in the equations of motion of the resulting force $\mathbf{R}$ and moment $\mathbf{M}$ of contact stresses $\sigma_{j0} = \sigma_{j3}|_{x_3=0}$. The contact force and moment can be computed as follows in the linear formulation

$$\mathbf{R} = R_j\mathbf{e}_j = \iint_{R^2} \sigma_{j0}\mathbf{e}_j dx_1 dx_2 = \iint_{\Omega} S_{j3}\mathbf{e}_j dx_1 dx_2 \tag{1.7}$$

$$\mathbf{M} = M_j\mathbf{e}_j = \mathbf{M}_0 - [\mathbf{u}_C,\mathbf{R}]$$

$$\mathbf{M}_0 = M_{j0}\mathbf{e}_j = \iint_{R^2} [\mathbf{r}_0,\ \sigma_{j0}\mathbf{e}_j] dx_1 dx_2 = \iint_{\Omega} [\mathbf{r}_0,\ S_{j3}\mathbf{e}_j] dx_1 dx_2$$

$$\sigma_{j0}(x_1,\ x_2,\ \tau) = S_{j3}(x_1,\ x_2,\ \tau)H(D_\sigma),\quad \mathbf{r}_0 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

$$D_\sigma = \operatorname{supp}\sigma_{j0} = \{(x_1, x_2, \tau) \mid (x_1, x_2) \in \Omega, \tau \geq 0\}$$

Here $H(D_\sigma)$ is the characteristic function of D_σ. In the case of boundary conditions (1.3) $R_1 = R_2 = M_3 = 0$, and the formulae for the remaining components can be simplified accordingly.

Relations (1.1), (1.5), (1.7), (1.2) and (1.3) along with the conditions determining Ω and the initial-value problem for the punch provide the mathematical formulation of the contact problem.

2. INFLUENCE FUNCTION

One possible approach to solving the transient contact problem in question is to use the influence functions for a half-space, which can be defined as follows: The displacements $u_n = G^1_{n,k}$ and stresses $\sigma_{ln} = \Gamma^1_{ln,k}$ corresponding to the boundary conditions

$$u_j\big|_{x_3=0} = \delta(x_1,\ x_2)\delta(\tau)\delta_{kj} \tag{2.1}$$

will be called the influence functions of the first kind (δ is the Diract delta function and δ_{kj} is the Kronecker delta).

Similarly, the displacements $u_n = G^2_{n,k3}$ and stresses $\sigma_{ln} = \Gamma^2_{ln,k3}$ defined by the boundary conditions

$$\sigma_{j3}\big|_{x_3=0} = \delta(x_1,\ x_2)\delta(\tau)\delta_{kj} \tag{2.2}$$

will be called the influence functions of the second kind, while $u_n = G^3_{n,3}$ and $\sigma_{ln} = \Gamma^3_{ln,3}$ will be referred to as the influence functions of the third kind if they satisfy the boundary conditions

$$u_3\big|_{x_3=0} = \delta(x_1,\ x_2)\delta(\tau),\quad \sigma_{12}\big|_{x_3=0} = \sigma_{13}\big|_{x_3=0} = 0 \tag{2.3}$$

Then the displacements and stresses at any point of the half-space can be written as convolutions of the influence functions with the appropriate functions defining the boundary conditions. Thus, on the plane $x_3 = 0$ we will have

$$(G^{m0}_{n,k} = G^m_{n,k}\big|_{x_3=0},\quad \Gamma^{m0}_{ln,k} = \Gamma^m_{ln,k}\big|_{x_3=0};\ m = 1,\ 2,\ 3)$$

$$u_{n0} = u_n\big|_{x_3=0} = u_{k0} * G^{10}_{n,k},\quad \sigma_{n0} = u_{k0} * \Gamma^{10}_{3n,k} \tag{2.4}$$

$$u_{n0} = \sigma_{k0} * G_{nk}^{20}, \quad \sigma_{n0} = \sigma_{k0} * \Gamma_{3n,k}^{20} \tag{2.5}$$

We denote by an asterisk the convolution with respect to the three variables x_1, x_2 and τ. The corresponding integrals are understood as regularized values, since the functions in (2.4) and (2.5) are in fact distributions.

The second set of convolutions in (2.4) and the first set in (2.5) are inverse operators to one another. The convolutions from the first set in (2.5) can be used as integral equations in contact problems (in the case of free slipping (1.3) $\sigma_{10} = \sigma_{20} = 0$). An example of such an approach is given, for example, in [7]. Convolutions of the following form correspond to the boundary conditions (2.3)

$$u_{n0} = u_{30} * G_{n,3}^{30}, \quad \sigma_{30} = u_{30} * \Gamma_{33,3}^{30} \tag{2.6}$$

To find the influence functions in problem (1.1), (1.4), and (2.1), ((2.2) or (2.3)) we apply the Fourier integral transform with respect to the spatial coordinates x_1, x_2 (we denote the transform by F and the parameters by p_1 and p_2) and the Laplace transform with respect to τ (L denotes the transform and s the parameter). Then Eqs (1.1) with (1.4) can be written in a matrix form in the image space of these transforms

$$\mathbf{D}_2\mathbf{U}'' - i\sum_{m=1}^{2} p_m \mathbf{D}_{1m}\mathbf{U}' - \sum_{j,m=1}^{2} p_j p_m \mathbf{D}_{0jm}\mathbf{U} = \gamma^2 s^2 \mathbf{U}$$

$$\mathbf{S} = \mathbf{D}_2\mathbf{U}' - i\sum_{m=1}^{2} p_m \mathbf{A}_{0m}\mathbf{U}, \quad \mathbf{S}_1 = \mathbf{B}_1\mathbf{U}' - i\sum_{m=1}^{2} \mathbf{B}_{0m}\mathbf{U} \tag{2.7}$$

$$\mathbf{X} = \mathbf{D}_2^0\mathbf{U}' + (\mathbf{E}_3 - i\sum_{m=1}^{2} p_m \mathbf{A}_{0m}^0)\mathbf{U}$$

$$\mathbf{U} = \begin{Vmatrix} u_1^{FL} \\ u_2^{FL} \\ u_3^{FL} \end{Vmatrix}, \quad \mathbf{S} = \begin{Vmatrix} \sigma_{13}^{FL} \\ \sigma_{23}^{FL} \\ \sigma_{33}^{FL} \end{Vmatrix}, \quad \mathbf{S}_1 = \begin{Vmatrix} \sigma_{11}^{FL} \\ \sigma_{12}^{FL} \\ \sigma_{22}^{FL} \end{Vmatrix}, \quad \mathbf{X} = \begin{Vmatrix} \sigma_{13}^{FL} \\ \sigma_{23}^{FL} \\ u_3^{FL} \end{Vmatrix}$$

$$\mathbf{E}_3 = \|\delta_{k3}\delta_{l3}\|_{3\times 3}, \quad \mathbf{E}^0 = \mathbf{E} - \mathbf{E}_3, \quad \mathbf{D}_2^0 = \mathbf{E}^0\mathbf{D}_2, \quad \mathbf{A}_{0m}^0 = \mathbf{E}^0\mathbf{A}_{0m}$$

Here $\mathbf{E}$ is the unit matrix, differentiation with respect to x_3 is denoted by a prime, and the 3×3 matrices $\mathbf{D}_2$, $\mathbf{D}_{1m}$, $\mathbf{D}_{0jm}$, $\mathbf{A}_{0m}$, $\mathbf{B}_1$ and $\mathbf{B}_{0m}$ are defined in terms of the elasticity constants of the medium

$$\mathbf{D}_2 = \|c_{m3n3}\|, \quad \mathbf{A}_{01} = \|c_{m3n1}\|, \quad \mathbf{A}_{02} = \|c_{m3n2}\|$$

$$\mathbf{D}_{011} = \|c_{m1n1}\|, \quad \mathbf{D}_{012} = \mathbf{D}_{021}^T = \|c_{m1n2}\|, \quad \mathbf{D}_{022} = \|c_{m2n2}\| \tag{2.8}$$

$$\mathbf{D}_{11} = \mathbf{A}_{01} + \mathbf{A}_{01}^T, \quad \mathbf{D}_{12} = \mathbf{A}_{02} + \mathbf{A}_{02}^T$$

$$\mathbf{B}_1 = \begin{Vmatrix} c_{1113} & c_{1123} & c_{1133} \\ c_{1213} & c_{1223} & c_{1233} \\ c_{2213} & c_{2223} & c_{2233} \end{Vmatrix}, \quad \mathbf{B}_{0m} = \begin{Vmatrix} c_{111m} & c_{112m} & c_{113m} \\ c_{121m} & c_{122m} & c_{123m} \\ c_{221m} & c_{222m} & c_{223m} \end{Vmatrix}$$

Analysing the characteristic equation of the system of differential equations

$$\sum_{n=0}^{6} a_n(p_1, p_2, s)\lambda^{6-m} = 0 \tag{2.9}$$

in (2.7), one can see that the coefficients $a_n(p_1, p_2, s)$ are homogeneous polynomials of degree n and the roots $\lambda_l(p_1, p_2, s)$ are homogeneous functions of order one with Re $\lambda_l < 0$ ($l = 1, 2, 3$). The corresponding system of algebraic equations implies that the eigenvectors $\boldsymbol{\lambda}_l^{(1)}(p_1, p_2, s)$ of the orthonormalized system are homogeneous functions of order zero.

The solution of (2.7) bounded at infinity has the form

$$\mathbf{U} = \sum_{l=1}^{3} C_l \boldsymbol{\gamma}_l^{(1)} f_l(x_3), \quad \mathbf{S} = \sum_{l=1}^{3} C_l \boldsymbol{\gamma}_l^{(2)} f_l(x_3)$$
$$\mathbf{X} = \sum_{l=1}^{3} C_l \boldsymbol{\gamma}_l^{(3)} f_l(x_3), \quad \mathbf{S}_1 = \sum_{l=1}^{3} C_l \boldsymbol{\gamma}_l^{(4)} f_l(x_3) \tag{2.10}$$
$$f_l(x_3) = \exp(\lambda_l x_3), \quad \boldsymbol{\gamma}_\alpha^{(2)} = (\mathbf{D}_2\lambda_\alpha - i\sum_{m=1}^{2} p_m \mathbf{A}_{0m})\boldsymbol{\gamma}_\alpha^{(1)}$$
$$\boldsymbol{\gamma}_\alpha^{(3)} = \mathbf{E}^0\boldsymbol{\gamma}_\alpha^{(2)} + \mathbf{E}_3\boldsymbol{\gamma}_\alpha^{(1)}, \quad \boldsymbol{\gamma}_\alpha^{(4)} = (\mathbf{B}_1\lambda_\alpha - i\sum_{m=1}^{2} p_m \mathbf{B}_{0m})\boldsymbol{\gamma}_\alpha^{(1)}$$
$$\boldsymbol{\gamma}_\alpha^{(j)} = \left\| \gamma_{1\alpha}^{(j)}, \quad \gamma_{2\alpha}^{(j)}, \quad \gamma_{3\alpha}^{(j)} \right\|^T$$

The constants C_l can be determined from the boundary conditions (2.1) or (2.2) or (2.3) in the image space ($[\delta(\tau)]^L = 1$, $[\delta(x_1, x_2)]^F = 1$). Then, for the influence functions of type j we obtain (there is no summation with respect to k)

$$\mathbf{U} = \left\| G_{1,\alpha}^{jFL}, \quad G_{2,\alpha}^{jFL}, \quad G_{3,\alpha}^{jFL} \right\|^T = \left|\boldsymbol{\Lambda}_j\right|^{-1} \sum_{l=1}^{3} (-1)^{k+l} M_{kl}^{(j)} \boldsymbol{\gamma}_l^{(1)} f_l(x_3)$$
$$\mathbf{S} = \left\| \Gamma_{13,\alpha}^{jFL}, \quad \Gamma_{23,\alpha}^{jFL}, \quad \Gamma_{33,\alpha}^{jFL} \right\|^T = \left|\boldsymbol{\Lambda}_j\right|^{-1} \sum_{l=1}^{3} (-1)^{k+l} M_{kl}^{(j)} \boldsymbol{\gamma}_l^{(2)} f_l(x_3) \tag{2.11}$$
$$\mathbf{S}_1 = \left\| \Gamma_{11,\alpha}^{jFL}, \quad \Gamma_{12,\alpha}^{jFL}, \quad \Gamma_{22,\alpha}^{jFL} \right\|^T = \left|\boldsymbol{\Lambda}_j\right|^{-1} \sum_{l=1}^{3} (-1)^{k+l} M_{kl}^{(j)} \boldsymbol{\gamma}_l^{(4)} f_l(x_3)$$
$$\boldsymbol{\Lambda}_n = \left\| \boldsymbol{\gamma}_1^{(n)}, \quad \boldsymbol{\gamma}_2^{(n)}, \quad \boldsymbol{\gamma}_3^{(n)} \right\|^T \quad (n = 1, 2), \quad \left|\boldsymbol{\Lambda}_1\right| = 1, \quad \boldsymbol{\Lambda}_3 = \mathbf{E}^0\boldsymbol{\Lambda}_2 + \mathbf{E}_3\boldsymbol{\Lambda}_1$$

Here $M_{kl}^{(j)}$ are the minors corresponding to the elements in the row k and column l of Λ_j and α is defined by $\alpha = k$ for $j = 1$, $\alpha = k3$ for $j = 2$, and $\alpha = k = 3$ for $j = 3$.

It is not possible to find analytic expressions for the original influence functions mainly because of the complexity of (2.9). No practical simplification of the computation of the origin functions can be achieved by considering anisotropic media with various symmetries, except for the well-known case of an isotropic medium, or by changing to a lower-dimensional (planar or axisymmetric) problem.

For the aforesaid forms of symmetry [8] (where the relation between the four-index symbols c_{ijkm} and the two-index symbols c_{ij} for the elasticity constants is also given) Eq. (2.9) becomes a bicubic equation.

$$\sum_{k=0}^{3} a_{2k}\lambda^{6-2k} = 0, \quad a_1 = a_3 = a_5 = 0 \tag{2.12}$$

and the following functional relations hold

second-order axis of symmetry x_3 or symmetry about the plane x_1x_2

$$a_{2k} = b_k(p_1^2, \ p_2^2, \ p_1p_2, \ s^2), \quad \lambda_l = -k_l(p_1^2, \ p_2^2, \ p_1p_2, \ s^2) \tag{2.13}$$

an orthotropic medium

$$a_{2k} = b_k(p_1^2, \ p_2^2, \ s^2), \quad \lambda_l = -k_l(p_1^2, \ p_2^2, \ s^2) \tag{2.14}$$

transversely isotropic medium with x_3 axis

$$a_{2k} = b_k(p_1^2 + p_2^2, \ s^2), \quad \lambda_l = -k_l(p_1^2 + p_2^2, \ s^2) \tag{2.15}$$

cubic symmetry

$$a_{2k} = b_k(p_1^2 + p_2^2,\ p_1^2 p_2^2,\ s^2),\quad \lambda_l = -k_l(p_1^2 + p_2^2,\ p_1^2 p_2^2,\ s^2) \tag{2.16}$$

an isotropic medium

$$-\lambda_1 = k_1 = \sqrt{p_1^2 + p_2^2 + \eta_1^2 s^2},\quad -\lambda_{2,3} = k_2 = \sqrt{p_1^2 + p_2^2 + \eta_2^2 s^2} \tag{2.17}$$

$$\eta_k = c_k / c_*,\quad c_{12} = \varkappa \gamma^2 / \eta_1^2,\quad c_{44} = \gamma^2 / \eta_2^2,\quad \varkappa = \lambda / (\lambda + 2\mu)$$

where c_k are the wave propagation velocities and λ and μ are the Lamé constants.

The transforms of the influence functions for the plane problem (when the tensor of elasticity constants is symmetric about the plane x_1x_3) can be obtained from (2.7) if we set $p_2 = 0, p_1 = p$, and $u_2 \equiv 0$. Then the characteristic polynomial (2.9) will be of degree four ($a_0 = a_1 = 0$). In the special cases of an orthotropic transversely isotropic medium or cubic symmetry the characteristic equation is biquadratic. Even though its roots can be written explicitly, it proves as difficult to obtain explicit formulae for the influence functions as in the general case of the plane problem.

Despite the complexity of the transforms (2.11) of the influence functions they can be used to establish a relation between the integral characteristics of the contact problem.

3. THE RELATION BETWEEN THE INTEGRAL CHARACTERISTICS

For problems with boundary conditions (1.2) or (1.3) the stresses σ_{j0} or σ_{30} have support Ω in the plane $x_3 = 0$. We will denote by Ω_u the analogous support for the displacements σ_{j0} ($\Omega \subset \Omega_u$). By the force integral characteristics we shall mean the integrals in (1.7), for which we retain the notation R_j and M_{0j} of the contact problem. The integral displacements U_j, velocities V_j, and their moments U_{mj}, V_{mj} ($j = 1, 2, 3; m = 1, 2$) of order one will be called the kinematic integral characteristics

$$U_{j0} = \iint_{R^2} u_{j0} dx_1 dx_2 = \iint_{\Omega_u} w_j dx_1 dx_2,\quad V_j = \iint_{R^2} \dot{u}_{j0} dx_1 dx_2$$

$$U_{mj} = \iint_{R^2} x_m u_{j0} dx_1 dx_2,\quad V_{mj} = \iint_{R^2} x_m \dot{u}_{j0} dx_1 dx_2 \tag{3.1}$$

$$u_{j0}(x_1,\ x_2,\ \tau) = w_j(x_1,\ x_2,\ \tau) H(D_u)$$

$$D_u = \operatorname{supp} u_{j0} = \{(x_1, x_2, \tau) \mid (x_1, x_2) \in \Omega_u, \tau \geq 0\}$$

We will establish a relation between the force and kinematic characteristics for the problem with boundary conditions (1.2). From (1.7), using the properties of the Fourier transform, we find in a similar way as in [2, 5, 6] that

$$R_j = \lim_{p_1, p_2 \to 0} \iint_{R^2} \sigma_{j0} e^{i(p_1 x_1 + p_2 x_2)} dx_1 dx_2 = \sigma_{j0}^F(0,\ 0,\ \tau)$$

$$M_{01} = \lim_{p_1, p_2 \to 0} (x_2 \sigma_{30})^F = -i \lim_{p_1, p_2 \to 0} \frac{\partial}{\partial p_2} \sigma_{30}^F \tag{3.2}$$

$$M_{02} = i \lim_{p_1, p_2 \to 0} \frac{\partial}{\partial p_1} \sigma_{30}^F,\quad M_{03} = -i \lim_{p_1, p_2 \to 0} \left[\frac{\partial}{\partial p_1} \sigma_{20}^F - \frac{\partial}{\partial p_2} \sigma_{10}^F \right]$$

By (2.4), to evaluate the limits in (3.2) it is necessary to know the limiting values of the transforms of the influence functions $\Gamma_{3n,k}^{10}$ and their derivatives. The eigenvalue problem for the system of differential equations in (2.7) has the following form for $p_1 = p_2 = 0$

$$|\mathbf{D}_2 - \zeta_l^2 \mathbf{E}| = 0,\quad (\mathbf{D}_2 - \zeta_l^2 \mathbf{E}) \boldsymbol{\gamma}_{l0} = 0,\quad \lambda_l \zeta_l = -\gamma s \tag{3.3}$$

$$\boldsymbol{\gamma}_{l0} = \boldsymbol{\gamma}_l^{(1)} \Big|_{p_1 = p_2 = 0} = \|\gamma_{1l0},\ \gamma_{2l0},\ \gamma_{3l0}\|^T,\quad \boldsymbol{\gamma}_{\alpha 0}^{(2)} = \boldsymbol{\gamma}_\alpha^{(2)} \Big|_{p_1 = p_2 = 0} = -\gamma s \zeta_\alpha \boldsymbol{\gamma}_{\alpha 0}$$

Since the matrix $\mathbf{D}_2$ is symmetric, its eigenvalues ζ_l^2 are real and positive. It can be shown that the numbers ζ_l^2 have a well-defined physical meaning, namely, $\zeta_l/\gamma = c_{3l}$ are the velocities of propagation of plane waves in the direction of $\mathbf{e}_3$.

Using the properties of integral transforms and formulae (2.4), (2.11) and (3.3), from (3.2) we find that ($\zeta_l > 0$; $l = 1, 2, 3$)

$$R_j^L(s) = -\gamma s \mu_{jk}^{(1)} u_{k0}^{FL}(0,\ 0,\ s) = -\gamma\mu_{jk}^{(1)} \iint_{R^2} s u_{k0}^L(x_1,\ x_2,\ s)dx_1dx_2 \tag{3.4}$$

$$\mu_{jk}^{(1)} = \Gamma_{j3,k}^{10FL}(0,\ 0,\ s) = |\Lambda_{1k}|$$

Here Λ_{1k} is the matrix obtained from $\Lambda_{10} = \Lambda_1|_{p1 = p2 = 0}$ by replacing row k by $\| \zeta_1\gamma_{j10}, \zeta_2\gamma_{j20}, \zeta_3\gamma_{j30} \|$. By a direct computation of the determinants $| \Lambda_{1k} |$ it can be shown that $\| \mu_{jk}^{(1)} \|$ is a symmetric matrix.

In the space of original functions (3.4) takes the form

$$R_j(\tau) = -\gamma\mu_{jk}^{(1)} V_k(\tau), \quad V_k(\tau) = \iint_{\Omega_u} \dot{w}_k dx_1dx_2 \tag{3.5}$$

The last equality can be obtained using the rule for differentiating generalized functions with bounded support and the absence of displacement discontinuities on the boundary ∂D_u of the support: $[w_k]_{\partial D_u} = 0$.

It is difficult to find the limiting values of the derivatives of the influence functions from (2.11). It is easier to find them by differentiating the system of equations in (2.7) with respect to p_n. Then, using boundary conditions (2.1), for $p_1 = p_2 = 0$ we obtain ($n = 1, 2$)

$$\begin{gathered} \mathbf{D}_2\mathbf{V}_{n0}'' - \gamma^2 s^2 \mathbf{V}_{n0} = i\mathbf{D}_{1n}\mathbf{U}_0, \quad \mathbf{Q}_{n0} = \mathbf{D}_2\mathbf{V}_{n0}' - i\mathbf{A}_{0n}\mathbf{U}_0 \\ \mathbf{V}_0\big|_{x_3=0} = \mathbf{V}_{00} = 0, \quad \mathbf{V}_{n0} = \mathbf{V}_n\big|_{p_1=p_2=0}, \quad \mathbf{U}_0 = \mathbf{U}\big|_{p_1=p_2=0} \\ \mathbf{Q}_{n0} = \mathbf{Q}_n\big|_{p_1=p_2=0}, \quad \mathbf{V}_n = \partial\mathbf{U}/\partial p_n, \quad \mathbf{Q}_n = \partial\mathbf{S}/\partial p_n \end{gathered} \tag{3.6}$$

The vector $\mathbf{U}_0$ is defined by (2.11).

Solving (3.6), we can find the required vector $\mathbf{Q}_{n00} = \mathbf{Q}_{n0}|_{x_3 = 0}$

$$\mathbf{Q}_{n0} = i(\mathbf{K}_n\mathbf{U}_0 + \tfrac{1}{2}x_3\mathbf{D}_{1n}\mathbf{U}_0') \tag{3.7}$$

$$\mathbf{Q}_{n00} = \left\| \frac{\partial}{\partial p_n}\Gamma_{13,k}^{jFL},\ \frac{\partial}{\partial p_n}\Gamma_{23,k}^{jFL},\ \frac{\partial}{\partial p_n}\Gamma_{33,k}^{jFL} \right\|^T_{p_1=p_2=x_3=0} = i\mathbf{K}_n\mathbf{U}_{00}$$

$$\mathbf{U}_{00} = \|\delta_{1k},\ \delta_{2k},\ \delta_{3k}\|^T, \quad \mathbf{K}_n = \tfrac{1}{2}(\mathbf{A}_{0n}^T - \mathbf{A}_{0n}) = \|\varkappa_{lm}^{(n)}\|_{3\times 3}$$

$$2\varkappa_{lm}^{(n)} = c_{lnm3} - c_{l3m3}$$

By the symmetry of the elasticity constants c_{ijkm}, the elements of the matrices $\mathbf{K}_n$ have the following properties

$$\begin{gathered} \varkappa_{lm}^{(n)} = -\varkappa_{ml}^{(n)}, \quad \varkappa_{lm}^{(3)} = \varkappa_{mm}^{(n)} = 0 \\ \varkappa_{lm}^{(n)} - \varkappa_{nm}^{(l)} = \varkappa_{ln}^{(m)}, \quad \varkappa_{l3}^{(n)} = \varkappa_{n3}^{(l)} \end{gathered} \tag{3.8}$$

Substituting (2.4) into (3.2) and using (3.7) and (3.8), we obtain the following formulae in the same way as (3.4) and (3.5) ($m = 1, 2$; $k = 1, 2, 3$)

$$M_{01}(\tau)=\varkappa_{3k}^{(2)}U_k(\tau)-\gamma\mu_{3k}^{(1)}V_{2k}(\tau)$$
$$M_{02}(\tau)=-\varkappa_{3k}^{(1)}U_k(\tau)-\gamma\mu_{3k}^{(1)}V_{1k}(\tau) \qquad (3.9)$$
$$M_{03}(\tau)=\varkappa_{21}^{(k)}U_k(\tau)-\gamma\mu_{2k}^{(1)}V_{1k}+\gamma\mu_{1k}^{(1)}V_{2k}(\tau)$$
$$V_{mk}=\iint\limits_{\Omega_u} x_m\dot{w}_k dx_1 dx_2$$

According to (3.8), the terms with $k = 3$ are equal to zero in the first sums. It follows that the moments M_{0j} are independent of the volume U_3 of the domain bounded by the deformed surface of the half-space and the plane $x_3 = 0$.

An argument similar to (3.2)–(3.9) leads to the following results for boundary conditions (1.3)

$$R_1=R_2=0, \quad R_3(\tau)=-\gamma\mu_{33}^{(3)}V_3(\tau)$$
$$M_{01}(\tau)=-\gamma\mu_{33}^{(3)}V_{23}(\tau), \quad M_{02}(\tau)=\gamma\mu_{33}^{(3)}V_{13}(\tau), \quad M_{03}=0 \qquad (3.10)$$
$$\mu_{33}^{(3)}=\sqrt{|\mathbf{D}_2|}/|\mathbf{\Lambda}_{33}|, \quad \mathbf{\Lambda}_{33}=\mathbf{E}^0\mathbf{\Lambda}_{23}+\mathbf{E}_3\mathbf{\Lambda}_{10}$$
$$\mathbf{\Lambda}_{23}=\|\zeta_1\gamma_{10}, \ \zeta_2\gamma_{20}, \ \zeta_3\gamma_{30}\|$$

Unlike (3.8), here the moments are independent of the integral displacements U_1 and U_2.

As follows from (3.4), (3.8) and (3.9), the integral forces and moments are linear combinations of the kinematic characteristics with coefficients depending only on the properties of the medium. According to (3.7), the non-zero coefficients $\varkappa_{lm}^{(n)}$ in (3.9) are defined by five pairs of differences of ten constants

$$2\varkappa_{31}^{(2)}=2\varkappa_{32}^{(1)}=c_{45}-c_{36}, \quad 2\varkappa_{32}^{(2)}=c_{44}-c_{23} \qquad (3.11)$$
$$2\varkappa_{31}^{(1)}=c_{55}-c_{13}, \quad 2\varkappa_{21}^{(1)}=c_{56}-c_{14}, \quad 2\varkappa_{21}^{(2)}=c_{25}-c_{46}$$

The coefficients $\mu_{lm}^{(j)}$ in (3.4), (3.9) and (3.10) depend on six different elements of the matrix $\mathbf{D}_2$: c_{55}, c_{45}, c_{35}, c_{44}, c_{34}, c_{33}. It follows that in the general case of anisotropy the contact forces and moments are independent of some elasticity constants: in the case of boundary conditions (1.3) they depend on six elements of the matrix $\mathbf{D}_2$, and in the case of (1.2) on 11 such elements (by (3.11) the above six elements are supplemented by c_{36}, c_{23}, c_{13}, $c_{56}-c_{14}$, $c_{25}-c_{46}$).

For elastic forces with symmetry one can find explicit formulae for the coefficients $\mu_{lm}^{(n)}$ and $\varkappa_{lm}^{(n)}$. Let us present the results for some cases of symmetry [8] (the first number in brackets indicates the total number of interdependent elasticity constants and the second one the number of constants in the formulae for the forces and moments).

Symmetry about the plane x_1x_3 (13–7)

$$\varkappa_{31}^{(2)}=\varkappa_{32}^{(1)}=\varkappa_{21}^{(1)}=\mu_{12}^{(1)}=\mu_{23}^{(1)}=0$$
$$\mu_{11}^{(1)}=\zeta_3-c_{35}^2(\zeta_3-\zeta_1)/\beta^2, \quad \mu_{13}^{(1)}=c_{35}(\zeta_3^2-c_{33})(\zeta_3-\zeta_1)$$
$$\mu_{33}^{(1)}=\zeta_1+\zeta_3-\mu_{11}^{(1)}, \quad \mu_{22}^{(1)}=\zeta_2, \quad \mu_{33}^{(3)}=\zeta_1\zeta_2/\mu_{11}^{(1)} \qquad (3.12)$$
$$\zeta_{1,3}^2=\frac{1}{2}\left(c_{33}+c_{55}\mp\sqrt{(c_{33}-c_{55})^2+4c_{35}^2}\right), \quad \zeta_3>\zeta_1$$
$$\beta^2=c_{35}^2+(\zeta_3^2-c_{33})^2$$

Second order axis of symmetry x_3 (13–7)

$$\varkappa_{21}^{(1)}=\varkappa_{21}^{(2)}=\mu_{13}^{(1)}=\mu_{23}^{(2)}=0$$
$$\mu_{11}^{(1)}=\zeta_2-c_{45}^2(\zeta_2-\zeta_1)/\beta^2, \quad \mu_{12}^{(1)}=c_{45}(\zeta_2^2-c_{44})(\zeta_2-\zeta_1)/\beta^2 \qquad (3.13)$$

$$\mu_{22}^{(1)} = \zeta_1 + \zeta_2 - \mu_{11}^{(1)}, \quad \mu_{33}^{(1)} = \mu_{33}^{(3)} = \zeta_3$$

$$\zeta_{1,2}^2 = \frac{1}{2}\left(c_{44} + c_{55} \mp \sqrt{(c_{44} - c_{55})^2 + 4c_{45}^2}\right), \quad \zeta_2 > \zeta_1$$

$$\zeta_3^2 = c_{33}, \quad \beta^2 = c_{45}^2 + (\zeta_1^2 - c_{33})^2$$

Orthotropic medium (9–5)

$$\varkappa_{31}^{(2)} = \varkappa_{32}^{(1)} = \varkappa_{21}^{(1)} = \varkappa_{21}^{(2)} = 0, \quad \mu_{\alpha k}^{(1)} = \zeta_\alpha \delta_{\alpha k} \tag{3.14}$$

$$\mu_{33}^{(3)} = \zeta_3, \quad \zeta_1^2 = c_{55}, \quad \zeta_2^2 = c_{44}, \quad \zeta_3^2 = c_{33}$$

The x_3 axis being an axis of symmetry of order three (7–3) *or four* (7–3), *transversely isotropic medium* (5–3), *or cubic symmetry* (3–3). These media are indistinguishable in terms of the formulae for the resulting forces and moments. Formulae (3.11) and (3.14) with $c_{55} = c_{44}$, $c_{13} = c_{23}$ remain valid.

Isotropic medium (2–2). In this case it is necessary to put $c_{33} = 2c_{44} + c_{13}$ in the computations for a transversely isotropic medium. Using (2.17), we have

$$\zeta_1 = \zeta_2 = \frac{\gamma}{\eta_2}, \quad \zeta_3 = \frac{\gamma}{\eta_1}, \quad \varkappa_{31}^{(1)} = \varkappa_{32}^{(2)} = \frac{\gamma^2}{2}\left(\frac{1}{\eta_2^2} - \frac{\varkappa}{\eta_1^2}\right) \tag{3.15}$$

Acoustic medium (1–1). It makes sense to consider only the boundary conditions (1.3) and the corresponding formulae (3.10), where $\mu_{33}^{(3)} = \gamma/\eta_1$.

4. RESULTANT REACTIONS

The relation between the contact forces (3.5), (3.9) and (3.10) can be applied directly to the contact problem only in the so-called supersonic case ($\Omega = \Omega_u$), which can be realized for a punch bounded by a smooth convex surface Π at the initial stages of the interaction. In this case the boundary $\partial\Omega$ of the contact domain can be defined precisely as the intersection of the boundary surface Π of the punch and the plane $x_3 = 0$.

The presence of a supersonic interaction stage for an anisotropic elastic medium is due to the bounds for the rate of change v_N of the boundary $\partial\Omega$ of the contact domain in the normal direction $\mathbf{N}_\Omega$

$$v_N \geqslant c_N \quad \forall (x_1, x_2) \in \partial\Omega, \quad c_N = \max(c_{1N}, c_{2N}, c_{3N}) \tag{4.1}$$

Here c_{kN} are the velocities of propagation of elastic waves in the direction of $\mathbf{N}_\Omega$. For an isotropic medium this inequality can be simplified accordingly [5].

First, we shall find the integral kinematic characteristics in (3.1) using a specific form of the displacements w_j in (1.5) and formulae (1.6) and (3.9)

$$U_j = S_j = S_{2,j} + u_{Cj} S, \quad V_j = \dot{u}_{Cj} S + \varepsilon_{jlm} \omega_l S_{2,m}$$

$$V_{kj} = u_{Ck} V_j + V_{2,kj}, \quad V_{2,kj} = \dot{u}_{Cj} S_{2,k} + \varepsilon_{jlm} \omega_l T_{2,km} \tag{4.2}$$

$$I_{kn} = I_{2,kn} + u_{Ck} S_{2,n} + u_{Cn} S_{2,k} + u_{Ck} u_{Cm} S$$

Here ε_{ijk} are the components of the Levi–Civita pseudotensor, and the coefficients of the kinematic parameters have the following geometric meaning: S is the contact domain area Ω; S_k and I_{km} (k, m = 1, 2) are its static moments and moments of inertia about the axes of the system of coordinates Ox_1x_2; S_{2k} and I_{2km} are the analogous geometrical characteristics in the system of coordinates $O_2z_1z_2$ obtained by a parallel translation of Ox_1x_2 to the point O_2, the projection of the centre of mass of the punch onto the plane $x_3 = S_{2,3}$ is the difference between the volume of the imbedded part G_0 of the punch and the volume of a cylinder with support Ω and height u_{C3}; $I_{23} = I_{32}$ and $I_{33}/2$ are the geometric static moments of G_0 with respect to the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, respectively.

Substituting (4.2) into (3.5), (3.9) and (3.10) and using (1.7), we obtain the following expressions for the resultant contact forces and moments *rigid contact*

$$R_j = -\gamma\mu_{jk}^{(1)}(\dot{u}_{Ck}S + \varepsilon_{jlm}S_{2,m}\omega_l)$$

$$M_1 = u_{C3}R_2 + \varkappa_{3k}^{(2)}S_k - \gamma\mu_{3k}^{(1)}V_{2,2k} \tag{4.3}$$

$$M_2 = -u_{C3}R_1 - \varkappa_{3k}^{(1)}S_k + \gamma\mu_{3k}^{(1)}V_{2,1k}$$

$$M_3 = \varkappa_{21}^{(k)}S_k - \gamma\mu_{2k}^{(1)}V_{2,1k} + \gamma\mu_{1k}^{(1)}V_{2,2k}$$

free slipping

$$R_1 = R_2 = 0, \quad R_3 = -\gamma\mu_{33}^{(3)}(\dot{u}_{C3}S + \omega_1 S_{2,2} - \omega_2 S_{2,1})$$

$$M_1 = -\gamma\mu_{33}^{(3)}V_{2,23}, \quad M_2 = \gamma\mu_{33}^{(3)}V_{2,13}, \quad M_3 = 0 \tag{4.4}$$

We observe that the expression for R_3 in (4.4) corresponds to that found in [6].

It follows that formulae (4.3) or (4.4) enable us to reduce the problem of determining the kinematic parameters of the punch to integrating a quasilinear system of ordinary differential equations in the supersonic case. To complete the problem one should construct formulae for the geometrical parameters S, $S_{2,k}$ and $I_{2,km}$ as functions of the linear angular displacements of the punch, which is easily done if the form of the surface Π is specified.

On the basis of (4.3) and (4.4) one can consider various cases of symmetry of an elastic medium in accordance with the discussion in Section 3 and investigate special cases of the contact problem such as plane-parallel and vertical motions of a punch or the plane problem. In the last case formulae (4.3) and (4.4) are the same as those obtained in [6] for an isotropic medium.

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